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Endomorphism Rings and Lattice Isomorphisms

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INTRODUCTION

A Baer ring is a ring in which every right (and left) annihilator ideal is generated by an idempotent. If B is a Baer ring, then the set of principal right ideals of B which are generated by idempotents forms a complete lattice which is anti-isomorphic to the lattice of principal left ideals generated by idempotents ([9], Proposition III 8.2). If ${}_RV$ is a vector space over a division ring, $\mathrm{Hom}_R(V, V)$ is a Baer ring. In trying to determine whether $B = \mathrm{Hom}_R(V, V)$ is a Baer ring for a general module ${}_RV$, a natural method to use is Baer's "Three-cornered Galois Theory," which establishes Galois connections between the lattice, $L(V)$, of submodules of V and the lattices $L(B.)$ and $L(.B)$ of right and left ideals of B, and consequently lattice isomorphisms and anti-isomorphisms between a certain sublattice, $\mathscr{C}$, of $L(V)$, consisting of the "closed" submodules of V and the lattices $\mathscr{A}_{\mathscr{R}}$ and $\mathscr{A}_{\mathscr{L}}$ of right and left annihilators in B. This method has been used in [11] and [10] for the case of a free module and in [6] for the case of a projective module which contains a unimodular element.

In both the free and the projective cases, two key steps are needed in order to establish the lattice isomorphisms: the first step consists in proving that the module V is a self-generator (i.e. $V\,\mathrm{Hom}_R(V, U) = U$ for every submodule U of V), while the second step consists in proving that the class of "closed" submodules under consideration (which, in [6], [10] and [11] happen to be the "dual-closed" submodules, i.e. the submodules which equal their own second annihilators with respect to the dual space $V^* = \mathrm{Hom}_R(V, R)$ of V) coincides with the class, $\mathscr{C}_a$, of "annihilator-closed" submodules of V, where the annihilator-closure operator is the one obtained from the Galois connection between $L(V)$ and $L(B.)$ which is given by Baer's three-cornered Galois Theory.

In the absence of "free-ness" or of a unimodular element, the module V need not be a self-generator; however, we show, in Section 2, that an "approximation" of self-generation is sufficient for the lattice (anti-) isomorphism between $(\mathscr{A}_{\mathscr{R}})$ $\mathscr{A}_{\mathscr{L}}$ and $\mathscr{C}_a$ to hold; and this allows us to escape from the "free" and the

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"unimodular" situations. This approximation, which we call c-self-generation (where $\varphi(U) = U^c$ is the closure operator under consideration), consists in the module V satisfying $[V \operatorname{Hom}_R(V, U)]^c = U$, for each c-closed submodule U. Two important and frequently considered closure operators are: the essential-closure operator, which we shall denote by $\varphi(U) = U^e$, and the dual-closure operator, $\varphi(U) = U^d$. We show, in Section 3, that any nonsingular module ${}_RV$, with the essential-closure operator, is an e-self-generating module, provided it satisfies the rather mild condition that $\operatorname{Hom}_R(V, U) \neq 0$ for every nonzero essentially-closed submodule U; and that any module ${}_RV$ with the dual-closure operator is d-self-generating, provided that V^* is T-faithful, where T is the trace (V, V^*) of V and we say, following Sandomierski ([8]), that a module X is T-faithful if $Xt \neq 0$ for each nonzero $t \in T$.

Whereas the first step of the method mentioned above can be generalized to modules which are not necessarily free or projective by requiring the modules to be c-self-generating, the second step must be resolved individually for each particular type of submodule and its closure operator. For a nonsingular module with the essential-closure operator, Theorem 3.5 gives necessary and sufficient conditions for the essentially closed submodules to coincide with the annihilator-closed ones, while for a general module with dual-closure, a sufficient condition is given in Proposition 3.6 for the dual-closed submodules to coincide with the annihilator-closed ones. Finally, applications of the three-cornered Galois Theory to Baer endomorphism rings are given in Theorems 3.7 and 3.8.

2. The Lattice Isomorphisms

Throughout this paper, R denotes an associative ring with 1, ${}_RV$ a unitary left R-module and $B = \operatorname{Hom}_R(V, V)$. The action of elements of B on V will be written on the right. The right (left) annihilator in B of a subset H of B will be denoted by $\mathscr{R}(H)(\mathscr{L}(H))$, while r and l will be used for annihilators in V of subsets of B, or in B of subsets of V, e.g. $l_V(H) = \{v \in V: vh = 0, \forall h \in H\}$ and $r_B(U) = \{b \in B: ub = 0 \forall u \in U\}$, $U \subseteq V$. Also, let $I(U) = \{b \in B: Vb \subseteq U\}$ and $UH = \{uh: u \in U \text{ and } h \in H\}$.

The following lemma is straightforward ([11], Lemma 1.1):

Lemma 2.1. *If $U \subseteq V$ and $J \subseteq B$, then*

(i) $VI(U) \subseteq U$.

(ii) $U \subseteq l_V r_B(U)$.

(iii) $I(U)\, r_B(U) = 0$.

(iv) $Il_V(J) = \mathscr{L}(J)$.

(v) $r_B(VJ) = \mathscr{R}(J)$. ∎

Let L be a complete lattice. A *closure operator* on L is a mapping $\varphi: L \to L$, written $\varphi(a) = a^c$, such that:

(c1) $a \leqslant b$ implies $a^c \leqslant b^c$;

(c2) $a \leqslant a^c$;

(c3) $(a^c)^c = a^c$.

An element a is *c-closed* if $a = a^c$. Given a closure operator, $\varphi(a) = a^c$, on a lattice L, we denote by $\mathscr{C}_c$ the set of c-closed elements of L.

Let L' be another complete lattice. A *Galois connection* between L and L' is a pair of mappings $\sigma: L \to L'$ and $\tau: L' \to L$ satisfying:

(1) $x_1 \leqslant x_2$ implies $\sigma(x_1) \geqslant \sigma(x_2)$ for $x_1, x_2 \in L$.

(2) $y_1 \leqslant y_2$ implies $\tau(y_1) \geqslant \tau(y_2)$ for $y_1, y_2 \in L'$.

(3) $x \leqslant \tau\sigma(x)$ and $y \leqslant \sigma\tau(y)$ for $x \in L, y \in L'$.

Given a Galois connection (σ, τ), it can be shown that $\sigma\tau\sigma(x) = \sigma(x)$ and $\tau\sigma\tau(y) = \tau(y)$ for $x \in L, y \in L'$, so that the maps $\tau\sigma$ and $\sigma\tau$ are closure operators on L and L', respectively. The closed elements in L are those which are of the form $\tau(y)$ for some $y \in L'$. σ and τ induce an anti-isomorphism between the corresponding lattices of closed elements ([9], pp. 76-78).

It is easily seen that $\mathscr{L}$ and $\mathscr{R}$ form a Galois connection between the lattice $L(B.)$ of right ideals of B and the lattice $L(.B)$ of left ideals of B, and that the mappings r_B and l_V form a Galois connection between the lattice $L(V)$ of submodules of V and the lattice $L(B.)$, giving the closure operators $r_B l_V$ and $l_V r_B$ on $L(V)$ and $L(B.)$ respectively. Let $\mathscr{C}_a = \{U \subseteq V: U = l_V r_B(U)\}$ be the collection of closed submodules of V with respect to the annihilator-closure operator $\varphi(U) = l_V r_B(U) = U^a$; the members of $\mathscr{C}_a$ will be referred to as the "annihilator-closed" or "a-closed" submodules of V.

Sandomierski ([7]) has called a module ${}_RV$ a *self-generator* if $VI(U) = U$ for every submodule U of V. In general, one only has $VI(U) \subset U$ (cf. Lemma 2.1 (i)); thus, in the presence of a closure operator, $\varphi(U) = U^c$, on $L(V)$, it is natural to approximate self-generation by requiring that $VI(U)$ and U have the same closure, at least when U is c-closed.

Definition 1. Let ${}_RV$ be a module with a closure operator, $\varphi(U) = U^c$, defined on its lattice of submodules, $L(V)$. V will be called a *c-self-generator* if $[VI(U)]^c = U$ for each c-closed $U \in L(V)$.

Definition 2. *Let* ${}_RV$ be a module with a closure operator, $\varphi(U) = U^c$. A subring D of $B = \mathrm{Hom}_R(V, V)$ will be called *c-continuous* if

$$X^c f \subseteq (Xf)^c \ \forall f \in D \qquad \text{and} \qquad X \in \mathscr{L}(V).$$

We shall show in the next section that any subring of B for a module (resp. a nonsingular module) with dual-closure (resp. essential-closure) is d-continuous (resp. e-continuous).

LEMMA 2.2. *Let ${}_RV$ be a module with a closure operator,* $\varphi(U) = U^c$; *if* $B = \mathrm{Hom}_R(V, V)$ *is c-continuous then*: (i) $r_B(X) = r_B(X^c)$ *for* $X \in \mathscr{L}(V)$, *and* (ii) $l_V(J) = [l_V(J)]^c$ *for* $J \subseteq B$.

Proof. Clearly, $X \subseteq X^c$ implies $r_B(X^c) \subseteq r_B(X)$; on the other hand, if $b \in r_B(X)$, then $X^cb \subseteq (Xb)^c = 0$, hence $b \in r_B(X^c)$, proving equality. Now let $U = l_V(J)$; if $x \in U^c$, then $xJ \subseteq U^cJ \subseteq (UJ)^c = 0$, therefore $x \in l_V(J) = U$, i.e. $U^c \subseteq U$, hence $U = U^c$. ∎

Remark. Since each member of $\mathscr{C}_a$ may be written in the form $l_V(J)$, for $J \subseteq B$, Lemma 2.2 (ii) implies that, for B c-continuous, $\mathscr{C}_a \subseteq \mathscr{C}_c$.

THEOREM 2.3. *Let ${}_RV$ be a c-self-generator and B be c-continuous.*

(a) *If* $U = U^c$, *then* $r_B(U) = \mathscr{R}[I(U)]$.

(b) *If* $J = \mathscr{R}(H)$, *for* $H \subseteq B$, *then* $J = r_Bl_V(J)$.

(c) *If* $H = \mathscr{L}(J)$, *for* $J \subseteq B$, *then* $l_V(J) = (VH)^c$ *and* $H = I[(VH)^c]$.

Proof. (a) Let $U = U^c$; then $r_B(U) = r_B[(VI(U))^c] = r_B[VI(U)]$ (by Lemma 2.2 (i)) $= \mathscr{R}[I(U)]$ (by Lemma 2.1 (v)).

(b) Let $J = \mathscr{R}(H)$, for $H \subseteq B$, and set $U = l_V(J)$. By Lemma 2.2 (ii), $U = U^c$, hence, by (a), $r_B(U) = \mathscr{R}[I(U)]$, or $r_Bl_V(J) = \mathscr{R}[Il_V(J)] = \mathscr{R}[\mathscr{L}(J)]$ (by Lemma 2.1 (iv)) $= \mathscr{R}\mathscr{L}\mathscr{R}(H) = \mathscr{R}(H) = J$.

(c) Let $H = \mathscr{L}(J)$, where $J \subseteq B$, and set $U = l_V(J)$. Since $U = U^c$ and V is a c-self-generator, $U = [VI(U)]^c$, or $l_V(J) = [VIl_V(J)]^c = [V\mathscr{L}(J)]^c = (VH)^c$. Consequently, $I[(VH)^c] = Il_V(J) = \mathscr{L}(J) = H$. ∎

Let

$$\mathscr{A}_{\mathscr{R}} = \{J \subseteq B\colon J \text{ is a right annihilator}\} \quad \text{and}$$
$$\mathscr{A}_{\mathscr{L}} = \{H \subseteq B\colon H \text{ is a left annihilator}\}.$$

THEOREM 2.4. *If ${}_RV$ is a c-self-generator and B is c-continuous then*: (a) *The maps* $\mathscr{A}_{\mathscr{R}} \to \mathscr{C}_a$ *by* $J \to l_V(J)$ *and* $\mathscr{C}_a \to \mathscr{A}_{\mathscr{R}}$ *by* $U \to r_B(U)$ *determine a lattice anti-isomorphism between* $\mathscr{C}_a$ *and* $\mathscr{A}_{\mathscr{R}}$. (b) *The maps* $\mathscr{A}_{\mathscr{L}} \to \mathscr{C}_a$ *by* $H \to (VH)^c$ *and* $\mathscr{C}_a \to \mathscr{A}_{\mathscr{L}}$ *by* $U \to I(U)$ *determine a lattice isomorphism between* $\mathscr{C}_a$ *and* $\mathscr{A}_{\mathscr{L}}$.

Proof. (a) Let $U \in \mathscr{C}_a$, then, since $\mathscr{C}_a \subseteq \mathscr{C}_c$, Theorem 2.3 (a) applies and $r_B(U) = \mathscr{R}[I(U)]$, i.e. $r_B(U) \in \mathscr{A}_{\mathscr{R}}$. Let $J \in \mathscr{A}_{\mathscr{R}}$, then, clearly, $l_Vr_B[l_V(J)] = l_V(J)$, so that $l_V(J) \in \mathscr{C}_a$. Since, by Theorem 2.3 (b), $J \in \mathscr{A}_{\mathscr{R}}$ implies $J = r_Bl_V(J)$, and, by definition, $U \in \mathscr{C}_a$ implies $U = l_Vr_B(U)$, it is now clear that r_B and l_V determine a lattice anti-isomorphism between $\mathscr{C}_a$ and $\mathscr{A}_{\mathscr{R}}$.

(b) If $U = l_V r_B(U)$ is in $\mathscr{C}_a$, then, using Lemma 2.1 (iv), $I(U) = Il_V r_B(U) = \mathscr{L}[r_B(U)] \in \mathscr{A}_{\mathscr{L}}$. Let $H = \mathscr{L}(J) \in \mathscr{A}_{\mathscr{L}}$; then, by Theorem 2.3 (c), $(VH)^c = l_V(J)$ is in $\mathscr{C}_a$.

Again by Theorem 2.3 (c), $H \in \mathscr{A}_{\mathscr{L}}$ implies $I[(VH)^c] = H$, and $U \in \mathscr{C}_a$ implies $[VI(U)]^c = [V\mathscr{L}(r_B(U))]^c = l_V r_B(U) = U$, showing that the given maps determine a lattice isomorphism. ∎

3. Examples

A module ${}_R V$ is an *essential extension* of a submodule U—written $U \subset' V$—if every nonzero submodule of V has nonzero intersection with U. One then says that U is *essential* in V. A submodule U of a module V is said to be *essentially closed* in V if U has no proper essential extensions in V. For $v \in V$, set $[U: v] = \{r \in R: rv \in U\}$; it is known that, if $U \subset' V$, then, for any nonzero $v \in V$, $[U: v] \subset' R$. The *singular submodule* $Z_R(V)$ of V is defined to be $\{v \in V: [0: v] \subset' R\}$. V is said to be *nonsingular* if $Z_R(V) = 0$. If $U \subset' V$, then V is nonsingular if and only if U is nonsingular. A ring will be called (left) nonsingular it its left regular representation is nonsingular. For details on essential extensions and nonsingular modules see [2] or [3]. If ${}_R V$ is a nonsingular module, then each submodule U of V has a unique maximal essential extension U^e in V, called the *essential closure* of U in V. This defines a closure operator, $\varphi(U) = U^e$, on $L(V)$.

Proposition 3.1. *If ${}_R V$ is a nonsingular module and $\varphi(U) = U^e$ is the essential closure of a submodule U, then $B = \mathrm{Hom}_R(V, V)$ is e-continuous.*

Proof. This is Lemma 3.2 of [5].

A module ${}_R V$ has been called *compressible* if, for each nonzero submodule X of V, $\mathrm{Hom}_R(V, X)$ contains a nonzero monomorphism ([1]). Generalizing this concept, we will call a module ${}_R V$ *retractable* if $\mathrm{Hom}_R(V, U) \neq 0$ for each nonzero submodule U of V. If ${}_R V$ has a closure operator, $\varphi(X) = X^c$, then ${}_R V$ will be called *c-retractable* if $\mathrm{Hom}_R(V, U) \neq 0$ for each nonzero c-closed submodule U. For example, any torsionless module over a semiprime ring is retractable (cf. [12] Proposition 1.2 (ii)).

Proposition 3.2. *If ${}_R V$ is a nonsingular e-retractable module, then V is e-self-generating.*

Proof. This proved in [5] (Theorem 3.4: (ii) $\Leftrightarrow$ (iii)).

Consider now a module ${}_R V$ in its standard Morita context, (R, V, V^*, B), that is, $V^* = \mathrm{Hom}_R(V, R)$, $B = \mathrm{Hom}_R(V, V)$, and ${}_R V_B$ and ${}_B V^*_R$ are, in a natural way, bimodules with an $R - R$ bimodule homomorphism(,): $V \otimes_B V^* \to R$ and a $B - B$ bimodule homomorphism [,]: $V^* \otimes_R V \to B$ given by $(v, g = vg$ (the evaluation map) and $(\ )[g, v] = (\ , g)\, v$ for $v \in V$ and $g \in V^*$; so that $(\ ,\ )$ and $[\ ,\ ]$ satisfy:

$$v_1[g_1, v_2] = (v_1, g_1)\, v_2 \text{ and } g_1(v_1, g_2) = [g_1, v_1]\, g_2, \forall v_1, v_2 \in V, g_1, g_2 \in V^*.$$

Note also that $(vb, g) = (v, bg)$ and $[gr, v] = [g, rv]\ \forall v \in V, b \in B,\ g \in V^*$ and $r \in R$. For $X \subseteq V$, define the annihilator of X in V^* by $X^\perp = \{g \in V^*\colon (X, g) = 0\}$, and for $Z \subseteq V^*$, define $Z = \{v \in V\colon (v, Z) = 0\}$. A submodule U of V will be called *dual-closed* if $U = {}^\perp U^\perp$. Clearly, $U \subseteq {}^\perp U^\perp$, ${}^\perp U^{\perp\perp} = U^\perp$ and $\varphi(U) = U^d = {}^\perp U^\perp$ defines a closure operator on $L(V)$.

PROPOSITION 3.3. *For any module ${}_RV$ with dual-closure, $\varphi(U) = U^d$, $B = \mathrm{Hom}_R(V, V)$ is d-continuous.*

Proof. To show continuity, we need to show $(X^d)\, b \subseteq (Xb)^d$, $\forall b \in B$. Let $y \in X^d = {}^\perp X^\perp$, so that $(y, X^\perp) = 0$; to show $yb \in (Xb)^d = {}^\perp(Xb)^\perp$, we must show $(yb, (Xb)^\perp) = 0$. Let $g \in (Xb)^\perp$, so that $(Xb, g) = 0$; then $(X, bg) = 0$, i.e. $bg \in X^\perp$. But this last implies $(y, bg) = 0$ and therefore $(yb, g) = 0$, as required. ∎

For a module ${}_RV$ with trace $T = (V, V^*)$, T was called V-faithful in [8] if $Tv \neq 0$ for each $0 \neq v \in V$. Analogously, we shall say that V^* is T-faithful if $V^*t \neq 0$ for each $0 \neq t \in T$.

PROPOSITION 3.4. *If V^* is T-faithful, then V is d-self-generating.*

Proof. Since $VI(U) \subseteq U$ implies $U^\perp \subseteq [VI(U)]^\perp$, it suffices to prove $[VI(U)]^\perp \subseteq U^\perp$. Let $g \in [VI(U)]^\perp$, i.e. $(VI(U), g) = 0$. Then, for any $b \in I(U)$, $(V, bg) = (Vb, g) = 0$, and therefore $bg = 0$. This means $I(U)\, g = 0$ $\forall g \in [VI(U)]^\perp$. Now, for any $u \in U$, $v[V^*, u] = (v, V^*)\, u \subseteq Ru$, i.e. $[V^*, u] \subseteq I(U)$, hence, by the preceding, $[V^*, u]\, g = 0$ $\forall g \in [VI(U)]^\perp$, or $V^*(u, g) = 0$. But since V^* is T-faithful, this implies $(u, g) = 0$, i.e. $(U, g) = 0$ for $g \in [VI(U)]^\perp$, or $g \in U^\perp$. ∎

We consider now the question of when the a-closure of a submodule coincides with its e-closure or with its d-closure. In the case of a nonsingular module with essential-closure, we have a complete answer in the following, where $E(V)$ denotes the injective hull of V.

THEOREM 3.5. *Let ${}_RV$ be a nonsingular module and let $\varphi(U) = U^e$ be the essential closure of a submodule U of V. Then the following are equivalent:*

(i) *$U^e = U^a$ for every submodule U of V.*

(ii) *$r_B(U) \neq 0$ for every $V \neq U \in \mathscr{C}_e$.*

(iii) *Every nonzero right ideal of* $\mathrm{Hom}_R(E(V), E(V))$ *has nonzero intersection with* $B = \mathrm{Hom}_R(V, V)$.

(iv) *$U = U^e \Rightarrow U = l_V(J)$, for some subset J of B.*

This is Theorem 3.5 of [5]. An example of a module satisfying the equivalent conditions of Theorem 3.5 is given by any finite-dimensional torsionless module over a ring which possesses a two-sided quotient ring ([4], Theorem 3.5).

For dual-closure, the following result holds:

PROPOSITION 3.6. *If ${}_RV$ is a T-faithful module, where $T = (V, V^*)$ is the trace of V, then ${}^{\perp}U^{\perp} = U^a$ for each submodule U of V.*

Proof. By d-continuity of $B = \mathrm{Hom}_R(V, V)$ (Proposition 3.3), $U = l_V(J)$ is dual-closed, so that ${}^{\perp}U^{\perp} \subseteq U^a$. Conversely, let $x \in U^a = l_V r_B(U)$. If $x \notin {}^{\perp}U^{\perp}$, then there is $g \in U^{\perp}$ such that $(x, g) \neq 0$. Since V is T-faithful, $(x, g)\, V \neq 0$, i.e. there is $0 \neq v \in V$ such that $(x, g)\, v = 0$. Consider $[g, v] = b \in B$: we have $Ub = U[g, v] = (U, g)\, v = 0$, so that $b \in r_B(U)$, and yet $xb = x[g, v] = (x, g)\, v \neq 0$, contradicting $x \in l_V r_B(U)$. Hence $x \in {}^{\perp}U^{\perp}$ and $U^a = {}^{\perp}U^{\perp}$. ∎

Note that any ${}_RV$, where R is semiprime, is T-faithful: given $0 \neq t = (y, f) \in T$, the semiprime-ness of R gives $0 \neq r \in R$ such that $(y, f)\, r(y, f) \neq 0$, and hence, in particular, $(y, f)\, ry \neq 0$. In fact, when R is semiprime, V^* is T-faithful also: if $0 \neq (y, f) \in T$, then $(y, f)\, r(y, f) \neq 0$, for $0 \neq r \in R$, also implies $fr(y, f) \neq 0$. Hence, for a module ${}_RV$ over a semiprime ring, there is a lattice (anti-)isomorphism between the (right) left annihilators in B and the dual-closed submodules of V.

Also, if V is T-faithful and T is V-faithful then V^* is T-faithful:

Proof.

$$\begin{aligned} V^*(v, w) = 0 &\Rightarrow [V^*(v, w), V] = 0 \Rightarrow [V^*, (v, w)\, V] = 0, \\ &\Rightarrow V[V^*, (v, w)\, V] = 0 \Rightarrow (V, V^*)(v, w)\, V = 0 \\ &\Rightarrow (v, w)\, V = 0, \qquad \text{since } T \text{ is } V\textit{-faithful} \\ &\Rightarrow (v, w) = 0, \quad \text{since } V \text{ is } T\text{-faithful.} \end{aligned}$$

When V is B-faithful, as is the case here, the condition "T is V-faithful" is equivalent to the "non-degeneracy condition: $[V^*, v] \neq 0$ for $v \neq 0$". Hence, in particular, if V is an R-faithful module satisfying the non-degeneracy condition, then there is a lattice (anti-)isomorphism between the (right) left annihilators in B and the dual-closed submodules of V.

A straightforward application of the methods and proofs of [11] yields the following:

THEOREM 3.7. *If R is semiprime or if ${}_RV$ is faithful and satisfies the non-degeneracy condition, then $B = \mathrm{Hom}_R(V, V)$ is a Baer ring if and only if every dual-closed submodule of V is a direct summand in V.*

THEOREM 3.8. *If ${}_RV$ is a finite-dimensional torsionless module over a ring R possessing a two-sided quotient ring, B is a Baer ring if and only if every essentially closed submodule of V is a direct summand in V.*

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